

Improved Separations of Regular Resolution from Clause Learning Proof Systems

Preliminary version. Comments appreciated.

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Abstract

We prove that the graph tautology formulas of Alekhovich, Johannsen, Pitassi, and Urquhart have polynomial size pool resolution refutations that use only input lemmas as learned clauses and without degenerate resolution inferences. We also prove that these graph tautology formulas can be refuted by polynomial size DPLL proofs with clause learning, even when restricted to greedy, unit-propagating DPLL search. We prove similar results for the guarded, xor-fied pebbling tautologies which Urquhart proved are hard for regular resolution.

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1 Introduction

The problem SAT of deciding the satisfiability of propositional CNF formulas is of great theoretical and practical interest. Even though it is NP-complete, industrial instances with hundreds of thousands variables are routinely solved by state of the art SAT solvers. Most of these solvers are based on the DPLL procedure [13] extended with clause learning, restarts, variable selection heuristics, and other techniques.

The basic DPLL procedure without clause learning is equivalent to tree-like resolution. The addition of clause learning makes DPLL considerably stronger. In fact, clause learning together with unlimited restarts is capable of simulating general resolution proofs [19]. However, the exact power of DPLL with clause learning but without restarts is unknown. This question is interesting not only for theoretical reasons, but also because of the potential for better understanding the practical performance of various refinements of DPLL with clause learning.

Beame, Kautz, and Sabharwal [3] gave the first theoretical analysis of DPLL with clause learning. Among other things, they noted that clause learning with restarts simulates general resolution. Their construction required the DPLL algorithm to ignore some contradictions, but this situation was rectified by Pipatsrisawat and Darwiche [19] who showed that SAT solvers which do not ignore contradictions can also simulate resolution. These techniques were also applied to learning bounded width clauses by Atserias et al. [2].

Beame et al. [3] also studied DPLL clause learning without restarts. Using a method of “proof trace extensions”, they were able to show that DPLL with clause learning and no restarts is strictly stronger than any “natural” proof system strictly weaker than resolution. Here, a *natural* proof system is one in which proofs do not increase in length substantially when variables are restricted to constants. The class of natural proof systems is known to include common proof systems such as tree-like or regular proofs. The proof trace method involves introducing extraneous variables and clauses, which have the effect of giving the clause learning DPLL algorithm more freedom in choosing decision variables for branching.

There have been two approaches to formalizing DPLL with clause learning as a static proof system rather than as a proof search algorithm. The first is pool resolution with a degenerate resolution inference, due originally to Van Gelder [23] and studied further by Hertel et al. [15]. Pool resolution requires proofs to have a depth-first regular traversal similarly to the search space of a DPLL algorithm. Degenerate resolution allows resolution infer-

ences in which one or both of the hypotheses may be lacking occurrences of the resolution literal. Van Gelder argued that pool resolution with degenerate resolution inferences simulates a wide range of DPLL algorithms with clause learning. He also gave a proof, based on [1], that pool resolution with degenerate inferences is stronger than regular resolution, using extraneous variables similar to proof trace extensions.

The second approach is due to Buss et al. [12] who introduced a “partially degenerate” resolution rule called w-resolution, and a proof system regWRTI based on w-resolution and clause learning of “input lemmas”. They proved that regWRTI exactly captures non-greedy DPLL with clause learning. By “non-greedy” is meant that contradictions may need to be ignored by the DPLL search.

Both [15] and [12] gave improved versions of the proof trace extension method so that the extraneous variables depend only on the set of clauses being refuted and not on resolution refutation of the clauses. The drawback remains, however, that the proof trace extension method gives contrived sets of clauses and contrived resolution refutations.

It remains open whether any of DPLL with clause learning, pool resolution (with or without degenerate inferences), or the regWRTI proof system can polynomially simulate general resolution. One approach to answering these questions is to try to separate pool resolution (say) from general resolution. So far, however, separation results are known only for the weaker system of regular resolution, based on work of Alekhnovich et al. [1], who gave an exponential separation between regular resolution and general resolution. Alekhnovich et al. [1] proved this separation for two families of tautologies, variants of the graph tautologies GT' and the “Stone” pebbling tautologies. Urquhart [22] subsequently gave a related separation using a different set of pebbling tautologies which he denoted Π_i .¹ In the present paper, we call the tautologies GT' the *guarded* graph tautologies, and henceforth denote them GGT instead of GT' ; their definition is given in Section 2. We define the formulas $GPeb^{k\oplus}(G)$ in Section 5; these are essentially the Π_i tautologies of Urquhart.

An obvious question is whether pool resolution (say) has polynomial size proofs of the GGT tautologies, the $GPeb^{k\oplus}$, or the Stone tautologies. The present paper resolves the first two questions by showing that pool resolution does indeed have polynomial size proofs of the graph tautologies GGT and

¹Huang and Yu [16] also gave a separation of regular resolution and general resolution, but only for a single set of clauses. Goerdt [14] gave a quasipolynomial separation of regular resolution and general resolution.

the pebbling tautologies $\text{GPeb}^{k\oplus}$. Our refutations avoid the use of extraneous variables in the style of proof trace extensions; furthermore, they use only the traditional resolution rule and do not require degenerate resolution inferences or w-resolution inferences. In addition, we use only learning of input clauses; thus, our refutations are also regWRTI proofs (and in fact regRTI proofs) in the terminology of [12]. As a corollary of the characterization of regWRTI by [12], the GGT principles and the $\text{GPeb}^{k\oplus}$ principles have polynomial size refutations that can be found by a DPLL algorithm with clause learning and without restarts (under the appropriate variable selection order).

The Stone principles have recently been shown to also have regRTI refutations by the second author and L. Kołodziejczyk [10]; however, their proof uses a rather different method than we use below. Thus, none of the three principles separate clause learning DPLL from full resolution. It is natural to speculate that perhaps pool resolution or regWRTI can simulate general resolution, or that DPLL with clause learning and without restarts can simulate general resolution. It is far from clear that this is true, but, if so, our methods and those of [10] may represent a step in this direction.

The outline of the paper is as follows. Section 2 begins with the definitions of resolution, degenerate resolution, and w-resolution, and then regular, tree, and pool resolution. After that, we define the graph tautologies GT_n and the guarded versions GGT_n , and state the main theorems about proofs of the GGT_n principles. Section 3 gives the proof of these theorems. Several ingredients are needed for the proof. The first idea is to try to follow the regular refutations of the graph tautology clauses GT_n as given by Stålmarck [21] and Bonet and Galesi [9]: however, these refutations cannot be used directly since the transitivity clauses of GT_n are “guarded” in the GGT_n clauses and this yields refutations which violate the regularity/pool property. So, the second idea is that the proof search process branches as needed to learn transitivity clauses. This generates additional clauses that must be proved: to handle these, we develop a notion of “bipartite partial order” and show that the refutations of [21, 9] can still be used in the presence of a bipartite partial order. The tricky part is to be sure that exactly the right set of clauses is derived by each subproof. Some straightforward bookkeeping shows that the resulting proof is polynomial size.

Section 4 discusses how to modify the refutations constructed in Section 3 so that they are “greedy” and “unit-propagating”. These conditions means that proofs cannot ignore contradictions, nor contradictions that can be obtained by unit propagation. The greedy and unit-propagating conditions correspond well to actual implemented DPLL proof search algorithms, since

they backtrack whenever a contradiction can be found by unit propagation. Section 4 concludes with an explicit description of a polynomial time DPLL clause learning algorithm for the GGT_n clauses.

Section 5 gives the pool resolution and regRTI refutations of the $\text{GPeb}^{k\oplus}$ principles. The proof mimics to a certain extent the methods of Section 3, but must also deal with the complications of xor-ification.

This paper is an expansion of an extended abstract [7] and an unpublished preprint [8] by the first two authors. These earlier versions included only the results for the GGT tautologies and did not consider the GPeb principles.

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2 Preliminaries and guarded graph tautologies

Propositional formulas are defined over a set of variables and the connectives \wedge , \vee and \neg . We use the notation \bar{x} to express the negation $\neg x$ of x . A *literal* is either a variable x or a negated variable \bar{x} . A *clause* C is a set of literals, interpreted as the disjunction of its members. The empty clause, \square , has truth value *False*. We shall only use formulas in *conjunctive normal form*, CNF; namely, a formula will be a set (conjunction) of clauses. We often use disjunction (\vee), union (\cup), and comma ($,$) interchangeably.

Definition The various forms of resolution take two clauses A and B called the *premises* and a literal x called the *resolution variable*, and produce a new clause C called the *resolvent*.

$$\frac{A \quad B}{C}$$

In all cases below, it is required that $\bar{x} \notin A$ and $x \notin B$. The different forms of resolution are:

Resolution rule. The hypotheses have the forms $A := A' \vee x$ and $B := B' \vee \bar{x}$. The resolvent C is $A' \vee B'$.

Degenerate resolution rule. [15, 23] If $x \in A$ and $\bar{x} \in B$, we apply the resolution rule to obtain C . If A contains x , and B doesn't contain \bar{x} , then the resolvent C is B . If A doesn't contain x , and B contains \bar{x} , then the resolvent C is A . If neither A nor B contains the literal x

or \bar{x} , then C is the lesser of A or B according to some tiebreaking ordering of clauses.

w-resolution rule. [12] From A and B as above, we infer $C := (A \setminus \{x\}) \vee (B \setminus \{\bar{x}\})$. If the literal $x \notin A$ (resp., $\bar{x} \notin B$), then it is called a *phantom literal* of A (resp., B).

Definition A *resolution derivation*, or *proof*, of a clause C from a CNF formula F is a sequence of clauses C_1, \dots, C_s such that $C = C_s$ and such that each clause from the sequence is either a clause from F or is the resolvent of two previous clauses. If the derived clause, C_s , is the empty clause, this is called a *resolution refutation* of F . The more general systems of degenerate and w-resolution refutations are defined similarly.

We represent a derivation as a directed acyclic graph (dag) on the vertices C_1, \dots, C_s , where each clause from F has out-degree 0, and all the other vertices from C_1, \dots, C_s have edges pointing to the two clauses from which they were derived. The empty clause has in-degree 0. We use the terms “proof” and “derivation” interchangeably.

Resolution is sound and complete in the refutational sense: a CNF formula F has a refutation if and only if F is unsatisfiable, that is, if and only if $\neg F$ is a tautology. Furthermore, if there is a derivation of a clause C from F , then C is a consequence of F ; that is, for every truth assignment σ , if σ satisfies F then it satisfies C . Conversely, if C is a consequence of F then there is a derivation of some $C' \subseteq C$ from F .

A resolution refutation is *regular* provided that, along any path in the directed acyclic graph, each variable is resolved on at most once. A resolution derivation of a clause C is *regular* provided that, in addition, no variable appearing in C is used as a resolution variable in the derivation. A refutation is *tree-like* if the underlying graph is a tree; that is, each occurrence of a clause occurring in the refutation is used at most once as a premise of an inference.

We next define a version of pool resolution, using the conventions of [12] who called this “tree-like regular resolution with lemmas” or “regRTL”. The idea is that clauses obtained previously in the proof can be used freely as learned lemmas. To be able to talk about clauses previously obtained, we need to define an ordering of clauses.

Definition Given a tree T , the *postorder* ordering $<_T$ of the nodes is defined as follows: if u is a node of T , v is a node in the subtree rooted at the left

child of u , and w is a node in the subtree rooted at the right child of u , then $v <_T w <_T u$.

Definition A *pool resolution* proof (also called a regRTL proof) from a set of initial clauses F is a resolution proof tree T that fulfills the following conditions: (a) each leaf is labeled with either a clause of F or a clause (called a “lemma”) that appears earlier in the tree in the $<_T$ ordering; (b) each internal node is labeled with a clause and a literal, and the clause is obtained by resolution from the clauses labeling the node’s children by resolving on the given literal; (c) the proof tree is regular; (d) the root is labeled with the conclusion clause. If the labeling of the root is the empty clause \square , the pool resolution proof is a pool refutation.

The notions of *degenerate pool resolution* proof and *pool w-resolution* proof are defined similarly, but allowing degenerate resolution or w-resolution inferences, respectively. The two papers [23, 15] defined pool resolution to be the degenerate pool resolution system, so our notion of pool resolution is more restrictive than theirs. Our definition is equivalent to the one in [11], however. It is also equivalent to the system regRTL defined in [12]. Pool w-resolution is the same as the system regWRTL of [12].

A “lemma” in clause (a) of the above definition is called an *input lemma* if it is derived by *input* subderivation, namely by a subderivation in which each inference has at least one hypothesis which is a member of F or is a lemma. The notion of input lemma was first introduced by [12]. In their terminology, a pool resolution proof which uses only input lemmas, is called a regRTI proof. Likewise a regWRTL proof that uses only input lemmas is called a regWRTI proof.

To understand the nomenclature; “reg” stands for “regular”, “W” for “w-resolution”, “RT” for “resolution tree”, “L” for lemma, and “I” for “input lemma”.

Next we define various graph tautologies, sometimes also called “ordering principles”. They will all use a size parameter $n > 1$, and variables $x_{i,j}$ with $i, j \in [n]$ and $i \neq j$, where $[n] = \{0, 1, 2, \dots, n-1\}$. A variable $x_{i,j}$ will intuitively represent the condition that $i < j$ with $<$ intended to be a total, linear order. We will thus always adopt the simplifying convention that $x_{i,j}$ and $\bar{x}_{j,i}$ are the identical literal, i.e., only the variables $x_{i,j}$ for $i < j$ actually exist, and $x_{j,i}$ for $j < i$ is just a notation for $\bar{x}_{i,j}$, and $\bar{x}_{j,i}$ stands for $x_{i,j}$. This identification makes no essential difference to the complexity of proofs of the tautologies, but it reduces the number of literals and clauses, and simplifies the definitions. In particular, it means there are no axioms for the antisymmetry or totality of $<$.

The following principle is based on the tautologies defined by Krishnamurthy [18]. These tautologies, or similar ones, have also been studied by [21, 9, 1, 4, 20, 24, 17].

Definition Let $n > 1$. Then GT_n is the following set of clauses involving the variables $x_{i,j}$, for $i, j \in [n]$ with $i \neq j$.

- (α_\emptyset) The clauses $\bigvee_{j \neq i} x_{j,i}$, for each value $i < n$.
- (γ_\emptyset) The *transitivity clauses* $T_{i,j,k} := \bar{x}_{i,j} \vee \bar{x}_{j,k} \vee \bar{x}_{k,i}$ for all distinct i, j, k in $[n]$.

Note that the clauses $T_{i,j,k}$, $T_{j,k,i}$ and $T_{k,i,j}$ are identical. For this reason Van Gelder [23] uses the name “no triangles” (NT) for a similar principle.

The next definition is from [1], who used the notation GT'_n . They used particular functions r and s for their lower bound proof, but since our upper bound proof does not depend on the details of r and s we leave them unspecified. We require that $r(i, j, k) \neq s(i, j, k)$ and that the set $\{r(i, j, k), s(i, j, k)\} \not\subseteq \{i, j, k\}$. In addition, w.l.o.g., $r(i, j, k) = r(j, k, i) = r(k, i, j)$, and similarly for s .

Definition Let $n \geq 1$, and let $r(i, j, k)$ and $s(i, j, k)$ be functions mapping $[n]^3 \rightarrow [n]$ as above. The *guarded graph tautology* formula GGT_n consists of the following clauses:

- (α_\emptyset) The clauses $\bigvee_{j \neq i} x_{j,i}$, for each value $i < n$.
- (γ'_\emptyset) The *guarded transitivity clauses* $T_{i,j,k} \vee x_{r,s}$ and $T_{i,j,k} \vee \bar{x}_{r,s}$, for all distinct i, j, k in $[n]$, where $r = r(i, j, k)$ and $s = s(i, j, k)$.

Note that the GGT_n clauses depend on the functions r and s ; this is suppressed in the notation. Our main result for the guarded graph tautologies is:

Theorem 1 *The guarded graph tautology formulas GGT_n have polynomial size pool (regRTL) resolution refutations.*

The proof of Theorem 1 will construct pool refutations in the form of regular tree-like refutations with lemmas. A key part of this is learning transitive closure clauses that are derived using resolution on the guarded transitivity clauses of GGT_n . A slightly modified construction, that uses a result from [12], gives instead tree-like regular resolution refutations with *input* lemmas. This will establish the following:

Theorem 2 *The guarded graph tautology formulas GGT_n have polynomial size, tree-like regular resolution refutations with input lemmas (regRTI refutations).*

A consequence of Theorem 2 is that the GGT_n clauses can be shown unsatisfiable by non-greedy polynomial size DPLL searches using clause learning. This follows via Theorem 5.6 of [12], since the refutations of GGT_n are regRTI, and hence regWRTI, proofs in the sense of [12].

However, as shown by Theorem 6 in Section 4, we can improve the constructions of Theorems 1 and 2 to show that the GGT_n principles can be refuted also by *greedy* and *unit-propagating* polynomial size DPLL searches with clause learning.

3 Guarded graph tautology refutations

The following theorem is an important ingredient of our upper bound proof.

Theorem 3 (Stålmarck [21]; Bonet-Galesi [9]; Van Gelder [24]) *The sets GT_n have regular resolution refutations P_n of polynomial size $O(n^3)$.*

We do not include a direct proof of Theorem 3 here, which can be found in [21, 9, 24]. The present paper uses the proofs P_n as a “black box”; the only property needed is that the P_n ’s are regular and polynomial size. Lemma 4 below is a direct generalization to Theorem 3; in fact, when specialized to the case of $\pi = \emptyset$, it is identical to Theorem 3.

The refutations P_n can be modified to give refutations of GGT_n by first deriving each transitive clause $T_{i,j,k}$ from the two guarded transitivity clauses of (γ'_\emptyset) . This however destroys the regularity property, and in fact no polynomial size regular refutations exist for GGT_n [1].

As usual, a *partial order* on $[n]$ is an antisymmetric, transitive relation binary relation on $[n]$. We shall be mostly interested in “partial specifications” of partial orders: partial specifications are not required to be transitive.

Definition A *partial specification*, τ , of a partial order is a set of ordered pairs $\tau \subseteq [n] \times [n]$ which are consistent with some (partial) order. The minimal partial order containing τ is the transitive closure of τ . We write $i \prec_\tau j$ to denote $\langle i, j \rangle \in \tau$, and write $i \prec_\tau^* j$ to denote that $\langle i, j \rangle$ is in the transitive closure of τ .

The τ -*minimal* elements are the i ’s such that $j \prec_\tau i$ does not hold for any j .

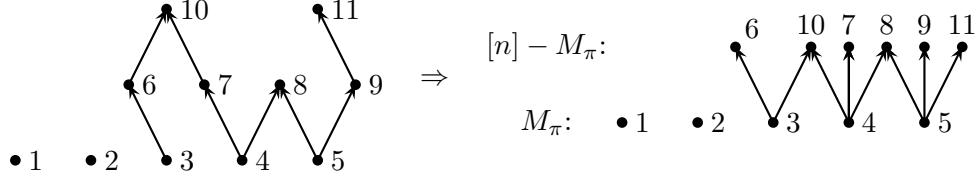


Figure 1: Example of a partial specification of a partial order (left) and the associated bipartite partial order (right).

We will be primarily interested in particular kinds of partial orders, called “bipartite” partial orders, that can be associated with partial orders. A bipartite partial order is a partial order that does not have any chain of inequalities $x \prec y \prec z$.

Definition A *bipartite partial order* is a binary relation π on $[n]$ such that the domain and range of π do not intersect. The set of π -minimal elements is denoted M_π .

The righthand side of Figure 1 shows an example. The bipartiteness of π arises from the fact that M_π and $[n] \setminus M_\pi$ partition $[n]$ into two sets. Note that if $i \prec_\pi j$, then $i \in M_\pi$ and $j \notin M_\pi$. In addition, M_π contains the isolated points of π .

Definition Let τ be a specification of a partial order. The bipartite partial order π that is *associated with* τ is defined by letting $i \prec_\pi j$ hold for precisely those i and j such that i is τ -minimal and $i \prec_\tau^* j$.

It is easy to check that the π associated with τ is in fact a bipartite partial order. The intuition is that π retains only the information about whether $i \prec_\tau^* j$ for minimal elements i , and forgets the ordering that τ imposes on non-minimal elements. Figure 1 shows an example of how to obtain a bipartite partial order from a partial specification.

We define the graph tautology $\text{GT}_{\pi,n}$ relative to π as follows.

Definition Let π be a bipartite partial order on $[n]$. Then $\text{GT}_{\pi,n}$ is the set of clauses containing:

- (α) The clauses $\bigvee_{j \neq i} x_{j,i}$, for each value $i \in M_\pi$.
- (β) The transitivity clauses $T_{i,j,k} := \bar{x}_{i,j} \vee \bar{x}_{j,k} \vee \bar{x}_{k,i}$ for all distinct i, j, k in M_π . (Vertices i, j, k' in Figure 2 show an example.)

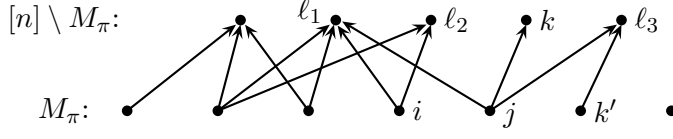


Figure 2: A bipartite partial order π is pictured, with the ordered pairs of π shown as directed edges. (For instance, $j \prec_\pi k$ holds.) The set M_π is the set of minimal vertices. The nodes i, j, k shown are an example of nodes used for a transitivity axiom $\bar{x}_{i,j} \vee \bar{x}_{j,k} \vee \bar{x}_{k,i}$ of type (γ) . The nodes i, j, k' are an example of the nodes for a transitivity axiom of type (β) .

- (γ) The transitivity clauses $T_{i,j,k}$ for all distinct i, j, k such that $i, j \in M_\pi$ and $i \not\prec_\pi k$ and $j \prec_\pi k$. (As shown in Figure 2.)

The set $\text{GT}_{\pi,n}$ is satisfiable if π is nonempty. As an example, there is the assignment that sets $x_{j,i}$ true for some fixed $j \notin M_\pi$ and every $i \in M_\pi$, and sets all other variables false. However, if π is applied as a restriction, then $\text{GT}_{\pi,n}$ becomes unsatisfiable. That is to say, there is no assignment which satisfies $\text{GT}_{\pi,n}$ and is consistent with π . This fact is proved by the regular derivation P_π described in the next lemma.

Definition For π a bipartite partial order, the clause $(\bigvee \bar{\pi})$ is defined by

$$\left(\bigvee \bar{\pi}\right) := \{\bar{x}_{i,j} : i \prec_\pi j\},$$

Lemma 4 Let π be a bipartite partial order on $[n]$. Then there is a regular derivation P_π of $(\bigvee \bar{\pi})$ from the set $\text{GT}_{\pi,n}$.

The only variables resolved on in P_π are the following: the variables $x_{i,j}$ such that $i, j \in M_\pi$, and the variables $x_{i,k}$ such that $k \notin M_\pi$, $i \in M_\pi$, and $i \not\prec_\pi k$.

Lemma 4 implies that if π is the bipartite partial order associated with a partial specification τ of a partial order, then the derivation P_π does not resolve on any literal whose value is set by τ . This is proved by noting that if $i \prec_\tau j$, then $j \notin M_\pi$.

Note that if π is empty, $M_\pi = [n]$ and there are no clauses of type (γ) . In this case, $\text{GT}_{\pi,n}$ is identical to GT_n , and P_π is the same as the refutation of GT_n of Theorem 3.

Proof By renumbering the vertices, we can assume w.l.o.g. that $M_\pi = \{0, \dots, m-1\}$. For each $k \geq m$, there is at least one value of j such that $j \prec_\pi k$: let J_k be an arbitrary such value j . Note $J_k < m$.

Fix $i \in M_\pi$; that is, $i < m$. Recall that the clause of type (α) in $\text{GT}_{\pi,n}$ for i is $\bigvee_{j \neq i} x_{j,i}$. We resolve this clause successively, for each $k \geq m$ such that $i \not\prec_\pi k$, against the clauses $T_{i,J_k,k}$ of type (γ)

$$\bar{x}_{i,J_k} \vee \bar{x}_{J_k,k} \vee \bar{x}_{k,i}$$

using resolution variables $x_{k,i}$. (Note that $J_k \neq i$ since $i \not\prec_\pi k$.) This yields a clause $T'_{i,m}$:

$$\bigvee_{\substack{k \geq m \\ i \not\prec_\pi k}} \bar{x}_{i,J_k} \vee \bigvee_{\substack{k \geq m \\ i \not\prec_\pi k}} \bar{x}_{J_k,k} \vee \bigvee_{\substack{k \geq m \\ i \prec_\pi k}} x_{k,i} \vee \bigvee_{\substack{k < m \\ k \neq i}} x_{k,i}.$$

The first two disjuncts shown above for $T'_{i,m}$ come from the side literals of the clauses $T_{i,J_k,k}$; the last two disjuncts come from the literals in $\bigvee_{j \neq i} x_{j,i}$ which were not resolved on. Since a literal \bar{x}_{i,J_k} is the same literal as $x_{J_k,i}$ and since $J_k < m$, the literals in the first disjunct are also contained in the fourth disjunct. Thus, eliminating duplicate literals, $T'_{i,m}$ is equal to the clause

$$\bigvee_{\substack{k \geq m \\ i \not\prec_\pi k}} \bar{x}_{J_k,k} \vee \bigvee_{\substack{k \geq m \\ i \prec_\pi k}} x_{k,i} \vee \bigvee_{\substack{k < m \\ k \neq i}} x_{k,i}.$$

Repeating this process, we obtain derivations of the clauses $T'_{i,m}$ for all $i < m$. The final disjuncts of these clauses, $\bigvee_{i \neq k < m} x_{k,i}$, are the same as the (α_\emptyset) clauses in GT_m . Thus, the clauses $T'_{i,m}$ give all (α_\emptyset) clauses of GT_m , but with literals $\bar{x}_{J_k,k}$ and $x_{k,i}$ added in as side literals. Moreover, the clauses of type (β) in $\text{GT}_{\pi,n}$ are exactly the transitivity clauses of GT_m . All these clauses can be combined exactly as in the refutation of GT_m described in Theorem 3, but carrying along extra side literals $\bar{x}_{J_k,k}$ and $x_{k,i}$, or equivalently carrying along literals $\bar{x}_{J_k,k}$ for $J_k \prec_\pi k$, and $\bar{x}_{i,k}$ for $i \prec_\pi k$. Since the refutation of GT_m uses all of its transitivity clauses and since each $\bar{x}_{J_k,k}$ literal is also one of the $\bar{x}_{i,k}$'s, this yields a resolution derivation P_π of the clause

$$\{\bar{x}_{i,k} : i \prec_\pi k\}.$$

This is the clause $(\bigvee \pi)$ as desired.

Finally, we observe that P_π is regular. To show this, note that the first parts of P_π deriving the clauses $T'_{i,m}$ are regular by construction, and they use resolution only on variables $x_{k,i}$ with $k \geq m$, $i < m$, and $i \not\prec_\pi k$. The remaining part of P_π is also regular by Theorem 3, and uses resolution only on variables $x_{i,j}$ with $i, j \leq m$. \square

Proof of Theorem 1. We will show how to construct a series of “LR partial refutations”, denoted R_0, R_1, R_2, \dots ; this process eventually terminates with a pool (regRTL) resolution refutation of GGT_n . The terminology “LR partial” indicates that the refutation is being constructed in left-to-right order, with the left part of the refutation properly formed, but with many of the remaining leaves being labeled with bipartite partial orders instead of with valid learned clauses or initial clauses from GGT_n . We first describe the construction of the pool refutation, and leave the size analysis to the end.

An LR partial refutation R is a tree with nodes labeled with clauses that form a correct pool resolution proof, except possibly at the leaves. Furthermore, it must satisfy the following conditions.

- a. R is a tree. The root is labeled with the empty clause. Each non-leaf node in R has a left child and right child; the clause labeling the node is derived by resolution from the clauses on its two children.
- b. For each clause C occurring in R , the clause C^+ and the set of ordered pairs $\tau(C)$ are defined by

$$C^+ := \{ \bar{x}_{i,j} : \bar{x}_{i,j} \text{ occurs in some clause on the branch from the root node of } R \text{ up to and including } C \},$$

and $\tau(C) = \{ \langle i, j \rangle : \bar{x}_{i,j} \in C^+ \}$. Note that $C \subseteq C^+$ holds by definition. In many cases, $\tau(C)$ will be a partial specification of a partial order, but this is not always true. For instance, if C is a transitivity axiom, then $\tau(C)$ has a 3-cycle and is not consistent as a specification of a partial order.

- d. Leaves are either “finished” or “unfinished”. Each finished leaf L is labeled with either a clause from GGT_n or a clause that occurs to the left of L in the postorder traversal of R .
- e. For an unfinished leaf labeled with clause C , the set $\tau(C)$ is a partial specification of a partial order. Furthermore, letting π be the bipartite partial order associated with $\tau(C)$, the clause C is equal to $(\bigvee \pi)$.

Property e. is particularly crucial and is novel to our construction. As shown below, each unfinished leaf, labeled with a clause $C = (\bigvee \pi)$, will be replaced by a derivation S . The derivation S often will be based on P_π , and thus might be expected to end with exactly the clause C ; however, some of

the resolution inferences needed for P_π might be disallowed by the regularity property of pool resolution proofs. This can mean that S will instead be a derivation of a clause C' such that $C \subseteq C' \subseteq C^+$. The condition $C' \subseteq C^+$ is required because any literal $x \in C' \setminus C$ will be handled by modifying the refutation R by propagating x downward in R until reaching a clause that already contains x . The condition $C' \subseteq C^+$ ensures that such a clause exists. The fact that $C' \supseteq C$ will mean that enough literals are present for the derivation to use only (non-degenerate) resolution inferences — by virtue of the fact that our constructions will pick C so that it contains the literals that must be present for use as resolution literals. The extra literals in $C' \setminus C$ will be handled by propagating them down the proof to where they are resolved on.

The construction begins by letting R_0 be the “empty” refutation, containing just the empty clause. Of course, this clause is an unfinished leaf, and $\tau(\emptyset) = \emptyset$. Thus R_0 is a valid LR partial refutation.

For the induction step, R_i has been constructed already. Let C be the leftmost unfinished clause in R_i . R_{i+1} will be formed by replacing C by a refutation S of some clause C' such that $C \subseteq C' \subseteq C^+$.

We need to describe the (LR partial) refutation S . Let π be the bipartite partial order associated with $\tau(C)$, and consider the derivation P_π from Lemma 4. Since C is $(\sqrt{\pi})$ by condition e., the final line of P_π is the clause C . The intuition is that we would like to let S be P_π . The first difficulty with this is that P_π is dag-like, and the LR -refutation is intended to be tree-like. This difficulty, however, can be circumvented by just expanding P_π , which is regular, into a tree-like regular derivation with lemmas by the simple expedient of using a depth-first traversal of P_π . The second, and more serious, difficulty is that P_π is a derivation from GT_n , not GGT_n . Namely, the derivation P_π uses the transitivity clauses of GT_n as initial clauses instead of the guarded transitivity clauses of GGT_n . The transitivity clauses $T_{i,j,k} := \bar{x}_{i,j} \vee \bar{x}_{j,k} \vee \bar{x}_{k,i}$ in P_π are handled one at a time as described below. We will use four separate constructions: in case (i), no change to P_π is required; cases (ii) and (iii) require small changes; and in the fourth case, the subproof P_π is abandoned in favor of “learning” the transitivity clause.

By the remark made after Lemma 4, no literal in C^+ is used as a resolution literal in P_π .

- (i) If a transitivity clause $T_{i,j,k}$ of P_π already appears earlier in R_i (that is, to the left of C), then it is already *learned*, and can be used freely in P_π .

In the remaining cases (ii)-(iv), the transitivity clause $T_{i,j,k}$ is not yet learned.

Let the guard variable for $T_{i,j,k}$ be $x_{r,s}$, so $r = r(i, j, k)$ and $s = s(i, j, k)$.

- (ii) Suppose case (i) does not apply and that the guard variable $x_{r,s}$ or its negation $\bar{x}_{r,s}$ is a member of C^+ . The guard variable thus is used as a resolution variable somewhere along the branch from the root to clause C . Then, as mentioned above, Lemma 4 implies that $x_{r,s}$ is not resolved on in P_π . Therefore, we can add the literal $x_{r,s}$ or $\bar{x}_{r,s}$ (respectively) to the clause $T_{i,j,k}$ and to every clause on any path below $T_{i,j,k}$ until reaching a clause that already contains that literal. This replaces $T_{i,j,k}$ with one of the initial clauses $T_{i,j,k} \vee x_{r,s}$ or $T_{i,j,k} \vee \bar{x}_{r,s}$ of GGT_n . By construction, it preserves the validity of the resolution inferences of R_i as well as the regularity property. Note this adds the literal $x_{r,s}$ or $\bar{x}_{r,s}$ to the final clause C' of the modified P_π . This maintains the property that $C \subseteq C' \subseteq C^+$.
- (iii) Suppose case (i) does not apply and that $x_{r,s}$ is not used as a resolution variable anywhere below $T_{i,j,k}$ in P_π and is not a member of C^+ . In this case, P_π is modified so as to derive the clause $T_{i,j,k}$ from the two GGT_n clauses $T_{i,j,k} \vee x_{r,s}$ and $T_{i,j,k} \vee \bar{x}_{r,s}$ by resolving on $x_{r,s}$. This maintains the regularity of the derivation. It also means that henceforth $T_{i,j,k}$ will be learned.

If all of the transitivity clauses in P_π can be handled by cases (i)-(iii), then we use P_π to define R_{i+1} . Namely, let P'_π be the derivation P_π as modified by the applications of cases (ii) and (iii). The derivation P'_π is regular and dag-like, so we can recast it as a tree-like derivation S with lemmas, by using a depth-first traversal of P'_π . The size of S is linear in the size of P'_π , since only input lemmas need to be repeated. The final line of S is the clause C' , namely C plus the literals introduced by case (ii). The derivation R_{i+1} is formed from R_i by replacing the clause C with the derivation S of C' , and then propagating each new literal $x \in C' \setminus C$ down towards the root of R_i , adding x to each clause below S until reaching a clause that already contains x . The derivation S contains no unfinished leaf, so R_{i+1} contains one fewer unfinished leaves than R_i .

On the other hand, if even one transitivity axiom $T_{i,j,k}$ in P_π is not covered by the above three cases, then case (iv) must be used instead. This introduces a completely different construction to form S :

- (iv) Let $T_{i,j,k}$ be any transitivity axiom in P_π that is not covered by cases (i)-(iii). In this case, the guard variable $x_{r,s}$ is used as a resolution variable in P_π somewhere below $T_{i,j,k}$; in general, this means we cannot use resolution on $x_{r,s}$ to derive $T_{i,j,k}$ while maintaining the desired

pool property. Hence, P_π is no longer used, and we instead will form S with a short left-branching path that “learns” $T_{i,j,k}$. This will generate two or three new unfinished leaf nodes. Since unfinished leaf nodes in a LR partial derivation must be labeled with clauses from bipartite partial orders, it is also necessary to attach short derivations to these unfinished leaf nodes to make the unfinished leaf clauses of S correspond correctly to bipartite partial orders. These unfinished leaf nodes are then kept in R_{i+1} to be handled at later stages.

There are separate constructions depending on whether $T_{i,j,k}$ is a clause of type (β) or (γ) ; details are given below.

First suppose $T_{i,j,k}$ is of type (γ) , and thus $\bar{x}_{j,k}$ appears in C . (Refer to Figure 2.) Let $x_{r,s}$ be the guard variable for the transitivity axiom $T_{i,j,k}$. The derivation S will have the form

$$\frac{\frac{\frac{\bar{x}_{i,j}, \bar{x}_{j,k}, \bar{x}_{k,i}, x_{r,s}}{\bar{x}_{i,j}, \bar{x}_{j,k}, \bar{x}_{k,i}} \quad \frac{\bar{x}_{i,j}, \bar{x}_{j,k}, \bar{x}_{k,i}, \bar{x}_{r,s}}{\bar{x}_{i,j}, \bar{x}_{j,k}, \bar{x}_{k,i}, \bar{x}_{r,s}} \quad S_1 \cdot \dots \cdot \dots}{\bar{x}_{i,j}, \bar{x}_{j,k}, \bar{\pi}_{-[jk;jR(i)]}} \quad \frac{S_2 \cdot \dots \cdot \dots}{\bar{x}_{j,i}, \bar{x}_{j,k}, \bar{\pi}_{-[jk;iR(j)]}}}{\bar{x}_{i,j}, \bar{x}_{j,k}, \bar{\pi}_{-[jk;jR(i)]} \quad \bar{x}_{j,i}, \bar{x}_{j,k}, \bar{\pi}_{-[jk;iR(j)]}}{\bar{x}_{j,k}, \bar{\pi}_{-[jk]}}$$

The notation $\bar{\pi}_{-[jk]}$ denotes the disjunction of the negations of the literals in π omitting the literal $\bar{x}_{j,k}$. We write “ $iR(j)$ ” to indicate literals $x_{i,\ell}$ such that $j \prec_\pi \ell$. (The “ $R(j)$ ” means “range of j ”.) Thus $\bar{\pi}_{-[jk;iR(j)]}$ denotes the clause containing the negations of the literals in π , omitting $\bar{x}_{j,k}$ and any literals $\bar{x}_{i,\ell}$ such that $j \prec_\pi \ell$. The clause $\bar{\pi}_{-[jk;jR(i)]}$ is defined similarly.

The upper leftmost inference of S is a resolution inference on the variable $x_{r,s}$. Since $T_{i,j,k}$ is not covered by either case (i) or (ii), the variable $x_{r,s}$ is not in C^+ . Thus, this use of $x_{r,s}$ as a resolution variable does not violate regularity. Furthermore, since $T_{i,j,k}$ is of type (γ) , we have $i \not\prec_{\tau(C)} j$, $j \not\prec_{\tau(C)} i$, $i \not\prec_{\tau(C)} k$, and $k \not\prec_{\tau(C)} i$. Thus the literals $x_{i,j}$ and $x_{i,k}$ are not in C^+ , so they also can be resolved on without violating regularity.

Let C_1 and C_2 be the final clauses of S_1 and S_2 , and let C_1^- be the clause below C_1 and above C . The set $\tau(C_2)$ is obtained by adding $\langle j, i \rangle$ to $\tau(C)$, and similarly $\tau(C_1^-)$ is $\tau(C)$ plus $\langle i, j \rangle$. Since $T_{i,j,k}$ is type (γ) , we have $i, j \in M_\pi$. Therefore, since $\tau(C)$ is a partial specification of a partial order, $\tau(C_2)$ and $\tau(C_1^-)$ are also both partial specifications of partial orders. Let π_2 and π_1 be the bipartite orders associated with these two partial specifications (respectively). We will form the subproof S_1 so that it contains the clause $(\bigvee \bar{\pi}_1)$ as its only unfinished clause. This will require

adding inferences in S_1 which add and remove the appropriate literals. The first step of this type already occurs in going up from C_1^- to C_1 since this has removed $\bar{x}_{j,k}$ and added $\bar{x}_{i,k}$, reflecting the fact that j is not π_1 -minimal and thus $x_{i,k} \in \pi_1$ but $x_{j,k} \notin \pi_1$. Similarly, we will form S_2 so that its only unfinished clause is $(\bigvee \bar{\pi}_2)$.

We first describe the subproof S_2 of S . The situation is pictured in Figure 3, which shows an extract from Figure 2: the edges shown in part (a) of the figure correspond to the literals present in the final line C_2 of S_2 . In particular, recall that the literals $\bar{x}_{i,\ell}$ such that $j \prec_\pi \ell$ are omitted from the last line of S_2 . (Correspondingly, the edge from i to ℓ_1 is omitted from Figure 3.) The last line C_2 of S_2 may not correspond to a bipartite partial order as it may not partition $[n]$ into minimal and non-minimal elements; thus, the last line of S_2 may not qualify to be an unfinished node of R_{i+1} . (An example of this in Figure 3(a) is that $j \prec_{\tau(C_2)} i \prec_{\tau(C_2)} \ell_2$, corresponding to $\bar{x}_{j,i}$ and \bar{x}_{i,ℓ_2} being in the last line of S_2 .) The bipartite partial order π_2 associated with $\tau(C_2)$ is equal to the bipartite partial order that agrees with π except that each $i \prec_\pi \ell$ condition is replaced with the condition $j \prec_{\pi_2} \ell$. (This is represented in Figure 3(b) by the fact that the edge from i to ℓ_2 has been replaced by the edge from j to ℓ_2 . Note that the vertex i is no longer a minimal element of π_2 ; that is, $i \notin M_{\pi_2}$.) We wish to form S_2 to be a regular derivation of the clause $\bar{x}_{j,i}, \bar{\pi}_{-[jk;iR(j)]}$ from the clause $(\bigvee \bar{\pi}_2)$.

The subproof of S_2 for replacing \bar{x}_{i,ℓ_2} in $\bar{\pi}$ with \bar{x}_{j,ℓ_2} in $\bar{\pi}_2$ is as follows, letting $\bar{\pi}^*$ be $\bar{\pi}_{-[jk;iR(j);i\ell_2]}$.

$$\frac{S'_2 \cdot \dots \cdot \quad \quad \quad \cdot \dots \cdot \text{rest of } S_2}{\frac{\bar{x}_{j,i}, \bar{x}_{i,\ell_2}, \bar{x}_{\ell_2,j} \quad \bar{x}_{j,k}, \bar{x}_{j,\ell_2}, \bar{x}_{j,i}, \bar{\pi}^*}{\bar{x}_{j,k}, \bar{x}_{i,\ell_2}, \bar{x}_{j,i}, \bar{\pi}^*}} \quad (1)$$

The part labeled “rest of S_2 ” will handle similarly the other literals ℓ such that $i \prec_\pi \ell$ and $j \not\prec_\pi \ell$. The final line of S'_2 is the transitivity axiom T_{j,i,ℓ_2} . This is a GT_n axiom, not a GGT_n axiom; however, it can be handled by the methods of cases (i)-(iii). Namely, if T_{j,i,ℓ_2} has already been learned by appearing somewhere to the left in R_i , then S'_2 is just this single clause. Otherwise, let the guard variable for T_{j,i,ℓ_2} be $x_{r',s'}$. If $x_{r',s'}$ is used as a resolution variable below T_{j,i,ℓ_2} , then replace T_{j,i,ℓ_2} with $T_{j,i,\ell_2} \vee x_{r',s'}$ or $T_{j,i,\ell_2} \vee \bar{x}_{r',s'}$, and propagate the $x_{r',s'}$ or $\bar{x}_{r',s'}$ to clauses down the branch leading to T_{j,i,ℓ_2} until reaching a clause that already contains that literal. Finally, if $x_{r',s'}$ has not been used as a resolution variable in R_i below C , then let S'_2 consist of a resolution inference deriving (and learning) T_{j,i,ℓ_2} from the clauses $T_{j,i,\ell_2}, x_{r',s'}$ and $T_{j,i,\ell_2}, \bar{x}_{r',s'}$.

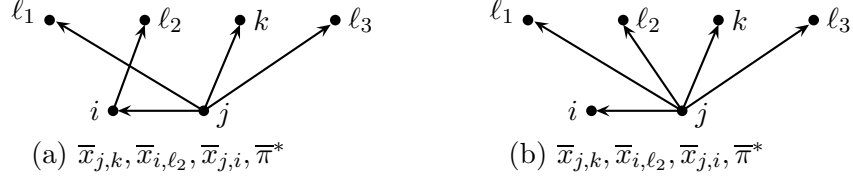


Figure 3: The partial orders for the fragment of S_2 shown in (1).

To complete the construction of S_2 , the inference (1) is repeated for each value of ℓ such that $i \prec_\pi \ell$ and $j \not\prec_\pi \ell$. The result is that S_2 has one unfinished leaf clause, and it is labeled with the clause $(\bigvee \bar{\pi}_2)$.

We next describe the subproof S_1 of S . The situation is shown in Figure 4. As in the formation of S_2 , the final clause C_1 in S_1 may need to be modified in order to correspond to the bipartite partial order π_1 which is associated with $\tau(C_1)$. First, note that the literal $\bar{x}_{j,k}$ is already replaced by $\bar{x}_{i,k}$ in the final clause of S_1 . The other change that is needed is that, for every ℓ such that $j \prec_\pi \ell$ and $i \not\prec_\pi \ell$, we must replace $\bar{x}_{j,\ell}$ with $\bar{x}_{i,\ell}$ since we have $j \not\prec_{\pi_1} \ell$ and $i \prec_{\pi_1} \ell$. Vertex ℓ_3 in Figure 4 is an example of a such a value ℓ . The ordering in the final clause of S_1 is shown in part (a), and the desired ordered pairs of π_1 are shown in part (b). Note that j is no longer a minimal element in π_1 .

The replacement of \bar{x}_{j,ℓ_3} with \bar{x}_{i,ℓ_3} is effected by the following inference, letting $\bar{\pi}^*$ now be $\bar{\pi}_{-[jk;jR(i);j\ell_3]}$.

$$\begin{array}{c}
 S'_1 \cdot \dots \cdot \dots \quad \quad \quad \cdot \dots \cdot \dots \text{ rest of } S_1 \\
 \hline
 \bar{x}_{i,j}, \bar{x}_{j,\ell_3}, \bar{x}_{\ell_3,i} \quad \quad \quad \bar{x}_{i,k}, \bar{x}_{i,\ell_3}, \bar{x}_{i,j}, \bar{\pi}^* \\
 \hline
 \bar{x}_{i,k}, \bar{x}_{j,\ell_3}, \bar{x}_{i,j}, \bar{\pi}^*
 \end{array} \tag{2}$$

The “rest of S_1 ” will handle similarly the other literals ℓ such that $j \prec_\pi \ell$ and $i \not\prec_\pi \ell$. Note that the final clause of S'_1 is the transitivity axiom T_{i,j,ℓ_3} . The subproof S'_1 is formed in exactly the same way that S'_2 was formed above. Namely, depending on the status of the guard variable $x_{r',s'}$ for T_{i,j,ℓ_3} , one of the following is done: (i) the clause T_{i,j,ℓ_3} is already learned and can be used as is, or (ii) one of $x_{r',s'}$ or $\bar{x}_{r',s'}$ is added to the clause and propagated down the proof, or (iii) the clause T_{i,j,ℓ_3} is inferred using resolution on $x_{r',s'}$ and becomes learned.

To complete the construction of S_1 , the inference (2) is repeated for each value of ℓ such that $j \prec_\pi \ell$ and $i \not\prec_\pi \ell$. The result is that S_1 has one unfinished leaf clause, and it corresponds to the bipartite partial order π_1 .

That completes the construction of the subproof S for the subcase of (iv) where $T_{i,j,k}$ is of type (γ) . Now suppose $T_{i,j,k}$ is of type (β) . (For instance,

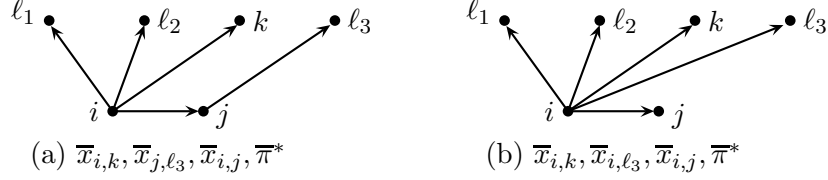


Figure 4: The partial orders for the fragment of S_1 shown in (2).

the values i, j, k' of Figure 2.) In this case the derivation S will have the form

$$\begin{array}{c}
 \frac{T_{i,j,k}, x_{r,s} \quad T_{i,j,k}, \bar{x}_{r,s}}{T_{i,j,k}} \quad \frac{S_3 \cdot \dots \cdot \cdot}{\bar{x}_{i,j}, \bar{x}_{i,k}, \bar{\pi}_{-[jR(i), kR(i \cup j)]}} \quad \frac{S_4 \cdot \dots \cdot \cdot}{\bar{x}_{i,j}, \bar{x}_{k,j}, \bar{\pi}_{-[jR(i \cap k)]}} \quad \frac{S_5 \cdot \dots \cdot \cdot}{\bar{x}_{j,i}, \bar{\pi}_{-[iR(j)]}} \\
 \hline
 \frac{\bar{x}_{i,j}, \bar{x}_{j,k}, \bar{\pi}_{-[jR(i), kR(i \cup j)]} \quad \bar{x}_{i,j}, \bar{x}_{k,j}, \bar{\pi}_{-[jR(i \cap k)]}}{\bar{x}_{i,j}, \bar{\pi}_{-[jR(i \cap k)]}} \quad \bar{\pi}
 \end{array}$$

where $x_{r,s}$ is the guard variable for $T_{i,j,k}$. We write $[\bar{\pi}_{-[jR(i \cap k)]}]$ to mean the negations of literals in π omitting any literal $\bar{x}_{j,\ell}$ such that both $i \prec_\pi \ell$ and $k \prec_\pi \ell$. Similarly, $\bar{\pi}_{-[jR(i), kR(i \cup j)]}$ indicates the negations of literals in π , omitting the literals $\bar{x}_{j,\ell}$ such that $i \prec_\pi \ell$ and the literals $\bar{x}_{k,\ell}$ such that either $i \prec_\pi \ell$ or $j \prec_\pi \ell$.

Note that the resolution on $x_{r,s}$ used to derive $T_{i,j,k}$ does not violate regularity, since otherwise $T_{i,j,k}$ would have been covered by case (ii). Likewise, the resolutions on $x_{i,j}$, $x_{i,k}$ and $x_{j,k}$ do not violate regularity since $T_{i,j,k}$ is of type (β) .

The subproof S_5 is formed exactly like the subproof S_2 above, with the exception that now the literal $\bar{x}_{j,k}$ is not present. Thus we omit the description of S_5 .

We next describe the construction of the subproof S_4 . Let C_4 be the final clause of S_4 ; it is easy to check that $\tau(C_4)$ is a partial specification of a partial order. As before, we must derive C_4 from the clause $(\bigvee \bar{\pi}_4)$ where π_4 is the bipartite partial order associated with the partial order $\tau(C_4)$. A typical situation is shown in Figure 5. As pictured there, it is necessary to add the literals $\bar{x}_{i,\ell}$ such that $j \prec_\pi \ell$ and $i \not\prec_\pi \ell$, while removing $\bar{x}_{j,\ell}$; examples of this are ℓ equal to ℓ_2 and ℓ_3 in Figure 5. At the same time, we must add the literals $\bar{x}_{k,\ell}$ such that $j \prec_\pi \ell$ and $k \not\prec_\pi \ell$, while removing $\bar{x}_{j,\ell}$; examples of this are ℓ equal to ℓ_1 and, again, ℓ_2 in the same figure.

For a vertex ℓ_3 such that $j \prec_\pi \ell_3$ and $k \prec_\pi \ell_3$ but $i \not\prec_\pi \ell_3$, this is done

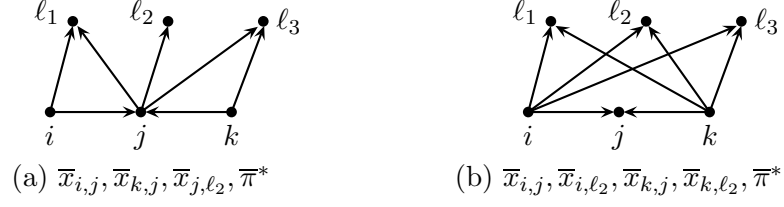


Figure 5: The partial orders as changed by S_4 .

similarly to the inferences (1) and (2) but without the side literal $\bar{x}_{j,k}$:

$$\frac{S'_4 \cdot \dots \cdot \bar{x}_{i,j}, \bar{x}_{j,\ell_3}, \bar{x}_{\ell_3,i} \quad \dots \cdot \text{rest of } S_4 \quad \bar{x}_{i,\ell_3}, \bar{x}_{k,j}, \bar{x}_{i,j}, \bar{\pi}^*}{\bar{x}_{j,\ell_3}, \bar{x}_{k,j}, \bar{x}_{i,j}, \bar{\pi}^*} \quad (3)$$

Here $\bar{\pi}^*$ is $\bar{\pi}_{-[jR(i \cap k); j\ell_3]}$. The transitivity axiom T_{i,j,ℓ_3} shown as the last line of S'_4 is handled exactly as before. This construction is repeated for all such ℓ_3 's.

The vertices ℓ_1 such that $j \prec_\pi \ell_1$ and $i \prec_\pi \ell_1$ but $k \not\prec_\pi \ell_1$ are handled in exactly the same way. (The side literals $\bar{\pi}^*$ change each time to reflect the literals that have already been replaced.)

Finally, consider a vertex ℓ_2 such that $i \not\prec_\pi \ell_2$ and $j \prec_\pi \ell_2$ and $k \not\prec_\pi \ell_2$. This is handled by the derivation

$$\frac{S''_4 \cdot \dots \cdot \bar{x}_{i,j}, \bar{x}_{j,\ell_2}, \bar{x}_{\ell_2,i} \quad \frac{S'''_4 \cdot \dots \cdot \bar{x}_{k,j}, \bar{x}_{j,\ell_2}, \bar{x}_{\ell_2,k} \quad \dots \cdot \text{rest of } S_4 \quad \bar{x}_{i,j}, \bar{x}_{i,\ell_2}, \bar{x}_{k,j}, \bar{x}_{k,\ell_2}, \bar{\pi}^*}{\bar{x}_{i,j}, \bar{x}_{i,\ell_2}, \bar{x}_{k,j}, \bar{x}_{j,\ell_2}, \bar{\pi}^*}}{\bar{x}_{i,j}, \bar{x}_{k,j}, \bar{x}_{j,\ell_2}, \bar{\pi}^*}$$

As before, the set $\bar{\pi}^*$ of side literals is changed to reflect the literals that have already been added and removed as S_4 is being created. The subproofs S''_4 and S'''_4 of the transitivity axioms T_{i,j,ℓ_2} and T_{k,j,ℓ_2} are handled exactly as before, depending on the status of their guard variables.

Finally, we describe how to form the subproof S_3 . For this, we must form the bipartite partial order π_3 which is associated with the partial order $\tau(C_3)$, where C_3 is the final clause of S_3 . To obtain $\bar{\pi}_3$, we need to add the literals $\bar{x}_{i,\ell}$ such that $i \not\prec_\pi \ell$ and such that either $j \prec_\pi \ell$ or $k \prec_\pi \ell$, while removing any literals $\bar{x}_{j,\ell}$ and $\bar{x}_{k,\ell}$. This is done by exactly the same construction used above in (3). The literals in $\bar{\pi}_{-[jR(i); kR(i \cup j)]}$ are exactly the literals needed to carry this out. The construction is quite similar to the above constructions, and we omit any further description.

That completes the description of how to construct the LR partial refutations R_i . The process stops once some R_i has no unfinished clauses. We claim that the process stops after polynomially many stages.

To prove this, recall that R_{i+1} is formed by handling the leftmost unfinished clause using one of cases (i)-(iv). In the first three cases, the unfinished clause is replaced by a derivation based on P_π for some bipartite order π . Since P_π has size $O(n^3)$, this means that the number of clauses in R_{i+1} is at most the number of clauses in R_i plus $O(n^3)$. Also, by construction, R_{i+1} has one fewer unfinished clauses than R_i . In case (iv) however, R_{i+1} is formed by adding up to $O(n)$ many clauses to R_i plus adding either two or three new unfinished leaf clauses. In addition, case (iv) always causes at least one transitivity axiom $T_{i,j,k}$ to be learned. Therefore, case (iv) can occur at most $2\binom{n}{3} = O(n^3)$ times. Consequently at most $3 \cdot 2\binom{n}{3} = O(n^3)$ many unfinished clauses are added throughout the entire process. It follows that the process stops with R_i having no unfinished clauses for some $i \leq 6\binom{n}{3} = O(n^3)$. Therefore there is a pool refutation of GGT_n with $O(n^6)$ lines. Since the GGT_n principle has $O(n^3)$ many clauses, the number of inferences in the refutation is bounded by a quadratic polynomial of the number of the clauses being refuted.

By inspection, each clause in the refutation contains $O(n^2)$ literals. This is because the largest clauses are those corresponding to (small modifications of) bipartite partial orders, and because bipartite partial orders can contain at most $O(n^2)$ many ordered pairs. Furthermore, the refutations P_n for the graph tautology GT_n contain only clauses of size $O(n^2)$.

Q.E.D. Theorem 1 □

Theorem 2 is proved with nearly the same construction. In fact, the only change needed is the construction of S from P'_π . Recall that in the proof of Theorem 1, the pool derivation S was formed by using a depth-first traversal of P'_π . This is not sufficient for Theorem 2, since now the derivation S must use only input lemmas. Instead, we use Theorem 3.3 of [12], which states that a (regular) dag-like resolution derivation can be transformed into a (regular) tree-like derivation with input lemmas. Forming S in this way from P'_π suffices for the proof of Theorem 2: the lemmas of S are either transitive closure axioms derived earlier in R_i or are derived by input subproofs earlier in the post-ordering of S . Since the transitive closure axioms that appeared earlier in R_i were derived by resolving two GGT_n axioms, the lemmas used in S are all input lemmas.

The transformation of Theorem 3.3 of [12] may multiply the size of the derivation by the depth of the original derivation. Since it is possible to form

the proofs P_π with depth $O(n)$, the overall size of the regRTI refutation is $O(n^7)$. This completes the proof of Theorem 2. \square

4 Greedy, unit-propagating DPLL with clause learning

This section discusses how the refutations in Theorems 1 and 2 can be modified so as to ensure that the refutations are greedy and unit-propagating.

Definition Let R be a tree-like regular w-resolution refutation with input lemmas. Let $\Gamma(C)$ be the set of clauses of Γ plus every clause $D <_R C$ in R that has been derived by an input subproof and thus is available as a learned clause to aid in the derivation of C .

The refutation R is *greedy and unit-propagating* provided that, for each clause C of R , if there is an input derivation from $\Gamma(C)$ of some clause $C' \subseteq C^+$ which does not resolve on any literal in C^+ , then C is derived in R by such a derivation.

Note that, as proved in [3], the condition that there is a input derivation from $\Gamma(C)$ of some $C' \subseteq C^+$ which does not resolve on C^+ literals is equivalent to the condition that if all literals of C^+ are set false then unit propagation yields a contradiction from $\Gamma(C)$. (In [3], these are called “trivial” proofs.) This justifies the terminology “unit-propagating”.

The definition of “greedy and unit-propagating” is actually a bit more restrictive than necessary, since DPLL algorithms may actually learn multiple clauses at once, and this can mean that C is not derived from a single input proof but rather from a combination of several input proofs as described in the proof of Theorem 5.1 in [12].

Theorem 5 *The guarded graph tautology formulas GGT_n have greedy, unit-propagating, polynomial size, tree-like, regular w-resolution refutations with input lemmas.*

Proof We indicate how to modify the proofs of Theorems 1 and 2. We again build tree-like LR partial refutations satisfying the same properties a.-e. as before, except now w-resolution inferences are permitted. Instead of being formed in distinct stages R_0, R_1, R_2, \dots , the w-resolution refutation R is constructed by one continuing process. This construction incorporates all of transformations (i)-(iv) and also incorporates the construction of Theorem 3.3 of [12].

At each point in the construction, we will be scanning the so-far constructed partial w-resolution refutation R in preorder, namely in depth-first, left-to-right order. That is to say, the construction recursively processes a node in the proof tree, then its left subtree, and then its right subtree. During most steps of the preorder scan, the partial refutation R is modified by changing the part that comes subsequently in the preorder, but the construction may also add and remove literals from clauses below the current clause C . When the preorder scan reaches a clause C that has an input derivation R' from $\Gamma(C)$ of some $C' \subseteq C$ that does not resolve on C^+ , then some such R' is inserted into R at that point. When the preorder scan reaches an unfinished leaf $C = C_0$, then a (possibly exponentially large) derivation P_π^* is added as its derivation. The construction continues processing R by scanning P_π^* in preorder, with the end result that either (1) P_π^* is successfully processed and reduced to only polynomial size or (2) the preorder scan of P_π^* reaches a transitivity clause $T_{i,j,k}$ of the type that triggered case (iv) of Theorem 1. In the latter case, the preorder scan backs up to the root clause C_0 of P_π^* , replaces P_π^* with the derivation S constructed in case (iv) of Theorem 1, and restarts the preorder scan at clause C_0 .

We describe the actions of the preorder scan in more detail. Initially, R is the “empty” derivation, with the empty clause as its only (unfinished) clause. A clause C encountered during the preorder scan of R is handled by one of the following.

- (i') Suppose that some $C' \subseteq C^+$ can be derived by an input derivation from $\Gamma(C)$ that does not resolve on any literals of C^+ . Fix any such $C' \subseteq C^+$, and replace the subderivation in R of the clause C with such a derivation of C' from $\Gamma(C)$. Any extra literals in $C' \setminus C$ are in C^+ and are propagated down until reaching a clause where they already appear, or occur as a phantom literal. There may also be literals in $C \setminus C'$: these literals are removed as necessary from clauses below C' in R to maintain the property of R containing correct w-resolution inferences. Note that this can convert resolution inferences into w-resolution inferences. The clause C' is now a learned clause.

Note that this case includes transitivity clauses $C = C' = T_{i,j,k}$ that satisfy the conditions of cases (i)-(iii) of Theorem 1

- (ii') If case (i') does not apply, and C is not a leaf node, then R is unchanged at this point and the depth-first traversal proceeds to the next clause.
- (iii') If C is an unfinished clause of the form $(\sqrt{\pi})$, let P_π be as before. Recall that no literal in C^+ is resolved on in P_π . Unwind the proof P_π

into a tree-like regular refutation P_π^* that is possibly exponentially big, and attach P_π^* to R as a proof of C . Mark the position of C by setting $C_0 = C$ in case it is necessary to later backtrack to C . Then continue the preorder scan by traversing into P_π^* .

- (iv') Otherwise, C is a leaf clause of the form $T_{i,j,k}$ and since case (i') does not apply, one of $T_{i,j,k}$'s guard literals x , namely $x = x_{r,s}$ or $x = \bar{x}_{r,s}$, is in C^+ . If C is *not* inside the most recently added P_π^* or if $x \in C_0^+$, then replace $T_{i,j,k}$ with $T_{i,j,k} \vee x$, and propagate the literal x downward in the refutation until reaching a clause where it appears as a literal or a phantom literal. Otherwise, the preorder scan backtracks to the root clause C_0 of P_π^* , and replaces P_π^* with the partial resolution refutation S formed in case (iv) of Theorem 1.

It is clear that this process eventually halts with a valid greedy, unit-propagating, tree-like w-resolution refutation. We claim that it also yields a polynomial size refutation. Consider what happens when a derivation P_π^* is inserted. If case (iv') is triggered, then the proof S is inserted in place of P_π^* , so the size of P_π^* does not matter. If case (iv') is not triggered, then, as in the proof of Theorem 3.3 of [12], the preorder scan of P_π^* modifies (the possibly exponentially large) P_π^* to have polynomial size. Indeed, as argued in [12], any clause C in P_π^* will occur at most d_C times in the modified version of P_π^* where d_C is the depth of the derivation of C in the original P_π . This is because C will have been learned by an input derivation once it has appeared no more than d_C times in the modified derivation P_π^* . This is proved by induction on d_C .

Consider the situation where S has just been inserted in place of P_π^* in case (iii'). The transitivity clause $T_{i,j,k}$ is not yet learned at this point, since otherwise case (i') would have applied. We claim, however, that $T_{i,j,k}$ is learned as S is traversed. To prove this, since $T_{i,j,k}$ is manifestly derived by an input derivation and since its guard literals $x_{r,s}$ and $\bar{x}_{r,s}$ do not appear in C_0^+ , it is enough to show that the clause $T_{i,j,k}$ is reached in the preorder traversal scan of S . This, however, is an immediate consequence of the fact that $T_{i,j,k}$ was reached in the preorder scan of P_π^* and triggered case (iv'), since if case (i') applies to $T_{i,j,k}$ or to any clause below $T_{i,j,k}$ in the preorder scan of S , then it certainly also applies $T_{i,j,k}$ or some clause below $T_{i,j,k}$ in the preorder scan of P_π^* .

The size of the final refutation R is bounded the same way as in the proof of Theorem 2, and this completes the proof of Theorem 5. \square

Theorem 6 *There are DPLL search procedures with clause learning which are greedy, unit-propagating, but do not use restarts, that refute the GGT_n clauses in polynomial time.*

We give a sketch of the proof. The construction for the proof of Theorem 5 requires only that the clauses $T_{i,j,k}$ are learned whenever possible, and does not depend on whether any other clauses are learned. This means that the following algorithm for DPLL search with clause learning will always succeed in finding a refutation of the GGT_n clauses: At each point, there is a partial assignment τ . The search algorithm must do one of the following:

- (1) If unit propagation yields a contradiction, then learn a clause $T_{i,j,k}$ if possible, and backtrack.
- (2) Otherwise, if there are any literals in the transitive closure of the bipartite partial order associated with τ which are not assigned a value, branch on one of these literals to set its value. (One of the true or false assignments yields an immediate conflict, and may allow learning a clause $T_{i,j,k}$.)
- (3) Otherwise, determine whether there is a clause $T_{i,j,k}$ which is used in the proof P_π whose guard literals are resolved on in P_π . (See Lemma 4.) If not, do a DPLL traversal of P_π , eventually backtracking from the assignment τ .
- (4) Otherwise, some clause $T_{i,j,k}$ blocks P_π from being traversed in polynomial time. Branch on its variables in the order given in the proof of Theorem 1. From this, learn the clause $T_{i,j,k}$.

5 Guarded, xor-ified, pebbling tautologies

This section gives polynomial size regRTI refutations for the guarded pebbling tautologies which Urquhart [22] proved require exponential size regular resolution proofs.

Definition A *pointed dag* $G = (V, E)$ is a directed acyclic graph with a single sink t such that every vertex in G has indegree either 0 or 2. The pebbling formula $\text{Peb}(G)$ for a pointed dag G is the unsatisfiable formula in the variables x_v for $v \in V$ consisting of the following clauses:

- (α) x_s , for every source $s \in V$,

- (β) $\bar{x}_u \vee \bar{x}_v \vee x_w$, for every vertex w with two (immediate) predecessors u and v ,
- (γ) \bar{x}_t , for t the sink vertex.

We next define Urquhart's "xor-ification" of a pebbling tautology clause. Xor-ification, for two variables, is due originally to Alekhovich and Razborov, see Ben-Sasson [5], and is similar to the "orification" used by [6]. The intuition for xor-ification is that each variable x_u is replaced by a set of clauses which express the k -fold exclusive or $x_{u,1} \oplus \cdots \oplus x_{u,k}$.

Definition Let $k > 0$, and x_u be a variable of the $\text{Peb}(G)$ principle. Let x_u^1 be x_u , and x_u^{-1} be its complement \bar{x}_u . Define $x_u^{k\oplus}$ to be the set of clauses of the form

$$x_{u,1}^{i_1} \vee x_{u,2}^{i_2} \vee \cdots \vee x_{u,k}^{i_k} \quad (4)$$

where an even number of the values i_j equal -1 (and the rest equal 1). Dually, define $\bar{x}_u^{k\oplus}$ to be the set of clauses of the form (4) with an odd number of the i_j equal to -1 . Note there are 2^{k-1} clauses in each of $x_u^{k\oplus}$ and $\bar{x}_u^{k\oplus}$. If C is a clause $C = z_1 \vee \cdots \vee z_\ell$, each z_i of the form x_u or \bar{x}_u , then $C^{k\oplus}$ is the set of clauses of the form

$$C_1 \vee C_2 \vee \cdots \vee C_\ell,$$

where each $C_i \in z_i^{k\oplus}$. There are $2^{(k-1)\ell}$ many clauses in $C^{k\oplus}$.

Definition The *xor-ified pebbling formula* $\text{Peb}^{k\oplus}(G)$ is the set of clauses $C^{k\oplus}$ for $C \in \text{Peb}(G)$.

Definition Let G be a pointed graph with n vertices and $k = k(n) > 0$. Let ρ be a function with domain the set of clauses of $\text{Peb}^{k\oplus}(G)$ and with range the set of variables $x_{u,i}$ of $\text{Peb}^{k\oplus}(G)$, such that, for all C , the variable $\rho(C)$ is not used in C . The *guarded xor-ified pebbling formula* $\text{GPeb}^{k\oplus}(G)$ is the set of clauses of the forms

$$C \vee \rho(C) \quad \text{and} \quad C \vee \overline{\rho(C)}$$

for $C \in \text{Peb}^{k\oplus}(G)$.

Note that, as in the case of GGT_n , the $\text{GPeb}^{k\oplus}(G)$ clauses depend on the choice of ρ ; again this is suppressed in the notation. For a dag G with n vertices, the formula $\text{GPeb}^{k\oplus}(G)$ consists of $O(2^{3k}n)$ clauses.

Our definitions of $\text{Peb}^{k\oplus}(G)$ and $\text{GPeb}^{k\oplus}(G)$ differ somewhat from Urquhart's, but these differences are inessential and make no difference to asymptotic proof sizes.

Of course, the $\text{Peb}^{k\oplus}(G)$ clauses are readily derivable from the $\text{GPeb}^{k\oplus}(G)$ clauses by resolving on the guard literals as given by ρ . There are simple polynomial size regular resolution refutations of the $\text{Peb}^{k\oplus}(G)$ principles; hence there are polynomial size, but not regular, resolution refutations of the $\text{GPeb}^{k\oplus}(G)$ principles. Indeed, Urquhart [22] proved that there are pointed graphs G with n vertices and values $k = k(n) = O(\log \log n)$, and functions ρ , such that regular resolution refutations of the $\text{GPeb}^{k\oplus}(G)$ clauses require size $2^{\Omega(n/((\log n)^2 \log \log n))}$.

We shall show that the $\text{GPeb}^{k\oplus}(G)$ principles have polynomial size proofs in regRTI and in pool resolution:

Theorem 7 *The guarded xor-ified pebbling formulas $\text{GPeb}^{k\oplus}(G)$ have polynomial size regRTI refutations, and thus polynomial size pool refutations.*

We conjecture that, analogously to Theorem 6, the $\text{GPeb}^{k\oplus}(G)$ principles can be shown unsatisfiable by polynomial size, greedy, unit propagating DPLL clause learning; however, we have not attempted to prove this.

We make some simple observations about working with xor-ified clauses before proving Theorem 7.

Lemma 8 *Let u be a vertex in G . There is a tree-like regular refutation of the clauses in $x_u^{k\oplus}$ and $\bar{x}_u^{k\oplus}$ with $2^k - 1$ resolution inferences. Its resolution variables are the variables $x_{u,i}$.*

Proof This is immediate by inspection: the refutation consists of resolving on the literals $x_{u,i}$ successively for $i = 1, 2, \dots, k$, giving a proof of height k . The leaf clauses of the proof are the members of $x_u^{k\oplus}$ and $\bar{x}_u^{k\oplus}$. \square

The refutation of Lemma 8 can be viewed as being the “ $k\oplus$ -translation” of the proof

$$\frac{x_u \quad \bar{x}_u}{\perp}$$

The next lemma describes a similar “ $k\oplus$ -translation” of a proof

$$\frac{C, x_u \quad D, \bar{x}_u}{C, D}$$

Lemma 9 *Let u be a vertex in G , and let C and D be clauses which do not contain either x_u and \bar{x}_u . Then each clause of $(C \vee D)^{k\oplus}$ has a tree-like regular derivation from the clauses in $(C \vee x_u)^{k\oplus}$ and $(D \vee \bar{x}_u)^{k\oplus}$ in which the variables used as resolution variables are exactly the variables $x_{u,i}$. This derivation has $2^k - 1$ resolution inferences, and 2^k leaf clauses.*

Proof Fix a clause E from $(C \vee D)^{k\oplus}$; we must describe its derivation from clauses in $(C \vee x_u)^{k\oplus}$ and $(D \vee \bar{x}_u)^{k\oplus}$. Let E_C be the subclause of E which is from $C^{k\oplus}$, and let E_D the subclause of E which is from $D^{k\oplus}$. If C and D have non-empty intersection, E_C and E_D are not disjoint; however, in any event, $E = E_C \cup E_D$.

Form the refutation from Lemma 8. Then add E_C to every leaf clause from x_u^\oplus , add E_D to every leaf clause from $\bar{x}_u^{k\oplus}$, and add E to every non-leaf clause. This gives the desired derivation of E . \square

Lemma 9 lets us further generalize the construction of $k\oplus$ -translations of proofs. As a typical example, the next lemma gives the $k\oplus$ -translation of the derivation

$$\frac{\frac{\bar{x}_u, \bar{x}_v, x_w}{\bar{x}_v, x_w} \quad x_u}{x_w} \quad x_v$$

Lemma 10 *Let w be a vertex of G , and u and v its predecessors. Then, each clause in $x_w^{k\oplus}$ has a dag-like resolution derivation P from the clauses in $x_u^{k\oplus}$, $x_v^{k\oplus}$, and $(\bar{x}_u \vee \bar{x}_v \vee x_w)^{k\oplus}$. This derivation contains $< 2^{2k}$ resolution inferences and resolves on the literals $x_{u,i}$ and $x_{v,i}$. In addition, the paths in P that lead to clauses in $x_v^{k\oplus}$ resolve on exactly the literals $x_{v,i}$.*

Lemma 10 follows by applying Lemma 9 twice. \square

It is important to note that the left-to-right order of the leaves of the derivation of Lemma 10 can be altered by changing the left-right order of hypotheses of resolution inferences. In particular, given any leaf clause D of a refutation P , we can order the hypotheses of the resolution inferences so that D is the leftmost leaf clause. This will be useful when D needs to be learned.

Definition $G|w$ is the induced subgraph of G with sink w and containing those vertices from which the vertex w is reachable. $G[w]$ is the graph resulting from G by making the vertex w a leaf by removing its incoming edges, and then removing those vertices from which the sink vertex of G is no longer reachable.

The vertex u is an *ancestor* of w if $u \neq w$ and $u \in G \upharpoonright w$. We call u and v *independent ancestors* of w provided u , v , and w are distinct and $u \in (G \upharpoonright w)[v]$ and $v \in (G \upharpoonright w)[u]$. In this case, we write $G[u, v]$ for $G[u][v] = G[v][u]$. Sometimes v , and possibly also u , may be missing or undefined; in these cases, $G[u, v]$ means just G if both u and v are undefined, and means $G[u]$ if u is present but v is undefined.

Note that it is possible for u and v to be independent ancestors of w , and also have u an ancestor of v or vice-versa.

The polynomial size regular resolution refutations of the $\text{Peb}^{k\oplus}(G)$ principles also apply to subgraphs such as $G \upharpoonright w$, $(G \upharpoonright w)[u]$ and $(G \upharpoonright w)[u, v]$. We can use these refutations to prove the following lemma. We write $\text{Peb}_{\alpha\beta}^{k\oplus}(G)$ to denote the $k\oplus$ -translations of $\text{Peb}(G)$ clauses of type (α) and (β) , omitting the clauses of type (γ) , and similarly for $\text{GPeb}_{\alpha\beta}^{k\oplus}(G)$.

- Lemma 11** (i) *Let w be a vertex of G . Then each clause in $x_w^{k\oplus}$ has a regular resolution derivation from the clauses $\text{Peb}_{\alpha\beta}^{k\oplus}(G \upharpoonright w)$. The derivation uses only resolution variables of the form $x_{a,i}$ with $a \in (G \upharpoonright w) \setminus \{w\}$.*
- (ii) *Let u and w be distinct vertices of G such that u is an ancestor of w . Then each clause in $(\bar{x}_u \vee x_w)^{k\oplus}$ has a regular resolution derivation from the clauses $\text{Peb}_{\alpha\beta}^{k\oplus}((G \upharpoonright w)[u])$. The derivation uses only resolution variables of the form $x_{a,i}$ for $a \in ((G \upharpoonright w)[u]) \setminus \{u, w\}$.*
- (iii) *Let u and v be independent ancestors of w . Then each clause of $(\bar{x}_u \vee \bar{x}_v \vee x_w)^{k\oplus}$ has a regular resolution derivation from the clauses $\text{Peb}_{\alpha\beta}^{k\oplus}((G \upharpoonright w)[u, v])$. The derivation uses only resolution variables of the form $x_{a,i}$ for $a \in ((G \upharpoonright w)[u, v]) \setminus \{u, v, w\}$.*

In all three cases, the derivation is dag-like and has size $O(2^{3k}n)$ and height $O(kn)$.

Proof We describe regular refutations from $\text{Peb}(G)$. Their $k\oplus$ -translations will give the regular derivations of Lemma 11.

For (i), there is an obvious regular dag-like, size $O(n)$, derivation P of x_w from the clauses (α) and (β) of the (non-xorified) principle $\text{Peb}(G)$; the derivation proceeds by resolving on literals x_u in a depth-first traversal of G . Forming the $k\oplus$ -translation $P^{k\oplus}$ of P forms the desired derivation for part (i) of any given clause of $x_w^{k\oplus}$. Since P has size $O(n)$ and clauses in P have at most three literals, $P^{k\oplus}$ has size $O(2^{3k}n)$ and height $O(kn)$. The limitation on which variables can be used as resolution variables follows by inspection; it is also a consequence of the fact that the refutation is regular.

The proofs of (ii) and (iii) are similar. \square

Proof (of Theorem 7.) We again construct a finite sequence of “LR partial refutations”, denoted R_0, R_1, R_2, \dots . This terminates after finitely many steps with the desired refutation R . Each LR partial refutation R_i will be a correct regRTI proof (and thus a correct pool resolution refutation); furthermore, it will satisfy the following conditions:

a. R_i is a tree of nodes labeled with clauses. The root is labeled with the empty clause. Each non-leaf node in R_i has a left child and a right child, and the clauses labeling these nodes form a valid resolution inference.

b. For each clause C in R_i , the clause C^+ is defined as before as

$$C^+ := \{\ell : \text{The literal } \ell \text{ occurs in some clause on the branch} \\ \text{from the root node of } R_i \text{ up to and including } C\}.$$

c. Each leaf of R_i is either “finished” or “unfinished”. Each finished node leaf L is labeled with either a clause from $\text{GPeb}^{k\oplus}(G)$ or with a clause that was derived by an input subderivation of R_i to the left of L in postorder. The input subderivation may not contain any unfinished leaves.

d. Each unfinished leaf is labeled with a clause $C \in E^{k\oplus}$ for a clause E such that one of the following three possibilities I.-III. holds. I. E is of the form x_w , and C^+ contains no literal $x_{a,i}$ for vertex $a \in (G \upharpoonright w) \setminus \{w\}$. II. E is of the form $\bar{x}_u \vee x_w$ with u an ancestor of w , and C^+ contains no literal $x_{a,i}$ for vertex $a \in (G \upharpoonright w)[u] \setminus \{u, w\}$. Or, III. E is of the form $\bar{x}_u \vee \bar{x}_v \vee x_w$ with u and v independent ancestors of w , and C^+ contains no literal $x_{a,i}$ for any vertex $a \in (G \upharpoonright w)[u, v] \setminus \{u, v, w\}$.

We introduce a new notational convention to describe (sub)clauses in R_i . For w a vertex in G , the notation W or W' denotes a clause in $x_w^{k\oplus}$, and \overline{W} or \overline{W}' to denotes a clause in $\bar{x}_w^{k\oplus}$. The notation \overline{W} or \overline{W}' in no way denotes the negation of W or W' ; instead, they are names of clauses, with the overline meant only to serve as a reminder of the semantic meaning.

The initial refutation R_0 is formed as follows. Let Q be the $k\oplus$ -translation of the inference

$$\frac{\overline{x}_t \quad x_t}{\perp}$$

as given by Lemma 8, where t is the sink of G . There are 2^k leaf clauses of Q : half of them are labeled with a clause $\overline{T} \in \overline{x}_t^{k\oplus}$ and the other half are labeled with a clause $T \in x_t^{k\oplus}$. Form R_0 from Q by replacing each leaf clause \overline{T} with a derivation

$$\frac{\overline{T}, \rho(\overline{T}) \quad \overline{T}, \overline{\rho(\overline{T})}}{\overline{T}}$$

These inferences are regular, since $\rho(\overline{T})$ is not an $x_{t,i}$. The other leaf clauses, of the form T , satisfy condition d. and are unfinished clauses in R_0 .

For the inductive step, the LR partial refutation R_i will be transformed into R_{i+1} . There are several cases to consider; the goal is to replace one unfinished leaf by a derivation containing only finished leaves, or to learn one more $\text{Peb}^{k\oplus}(G)$ clause while adding only polynomially many more unfinished leaves.

Consider the leftmost unfinished leaf of R_i . By condition d., its clause C will have one of the forms W , or \overline{U}, W , or $\overline{U}, \overline{V}, W$ where $\overline{U} \in \overline{x}_u^{k\oplus}$, $\overline{V} \in \overline{x}_v^{k\oplus}$, and $W \in x_w^{k\oplus}$. By Lemma 11, there is a dag-like regular refutation P of C from the clauses of $\text{Peb}_{\alpha\beta}^{k\oplus}((G \upharpoonright w)[u, v])$. We wish to convert P into a derivation from the clauses of $\text{GPeb}_{\alpha\beta}^{k\oplus}((G \upharpoonright w)[u, v])$ and the already learned clauses of R_i . Consider each leaf clause D of P . Then D is a $k\oplus$ -translation of a clause in $\text{Peb}_{\alpha\beta}^{k\oplus}((G \upharpoonright w)[u, v])$. As in the proof of Theorem 1, there are four cases to consider:

- (i) If the clause D is already learned as an input lemma in R_i to the left of C , then D may be used in P as is.

For the remaining cases, assume D has not been learned as an input lemma.

- (ii) Let $y = \rho(D)$. If either y or \overline{y} is a member of C^+ , then add that literal to D and every clause on the path below D until reaching the first clause where it appears. This replaces D with the $\text{GPeb}_{\alpha\beta}^{k\oplus}((G \upharpoonright w)[u, v])$ clause $D \vee y$ or $D \vee \overline{y}$.
- (iii) Suppose cases (i) and (ii) do not apply and that y is not used as a resolution variable below D . In this case, replace D by a resolution inference deriving D from $D \vee y$ and $D \vee \overline{y}$. This preserves the regularity of the derivation. It also makes D a learned clause.

It is possible that C itself is a $\text{Peb}_{\alpha\beta}^{k\oplus}(G)$ clause. If so, then $C = D$ and P is the trivial derivation containing only C , and one of cases (i)-(iii) holds.

If all leaf clauses D of P can be treated by cases (i)-(iii), then we have successfully transformed P into a (still dag-like) derivation P' which satisfies regularity and in which leaf clauses are from $\text{GPeb}^{k\oplus}(G)$ or already learned as input lemmas in R_i . By Theorem 3.3 of [12], P' can be converted in a regRTI proof P'' of the same conclusion as P , preserving the regularity conditions, and with the size of P' bounded by twice the product of the size of P and the height of P . Therefore, the size of P'' is $O((2^{3k}n)(kn)) = O(k2^{3k}n^2)$. Form R_{i+1} by replacing the clause C in R_i with the derivation P'' . R_{i+1} satisfies conditions a.-d., and has one fewer unfinished clauses than R_i .

However, if even one leaf clause D of P fails cases (i)-(iii), then the entire subderivation P is abandoned, and we chose some leaf clause D of P to be learned such that D does not fall into cases (i)-(iii).

The leaf clause D is the $k\oplus$ -translation of an (α) or (β) clause of $\text{Peb}^{k\oplus}(G)$ and thus either has the form $E \in x_e^\oplus$ for some source e in G or has the form $\overline{A}, \overline{B}, E$ where $\overline{A} \in \overline{x}_a^{k\oplus}$, $\overline{B} \in \overline{x}_b^{k\oplus}$, and $E \in x_e^{k\oplus}$ for a, b , and e vertices in $G \setminus w$ with a and b the two predecessors of e in G . Without loss of generality, b is not an ancestor of a in G ; otherwise interchange a and b . The construction now splits into three cases I., II., and III. depending on the form of C . These three cases each split into two subcases depending whether D is E or is $\overline{A}, \overline{B}, E$.

I. In the first case, C is equal to just W . Recall that W is a clause in $x_w^{k\oplus}$. First suppose D is equal to E . Consider the derivation structure

$$\frac{x_e \quad \overline{x}_e, W}{W} \quad (5)$$

Note that (5) contains a blend of variables from $\text{Peb}(G)$ (non-xorified) and from $\text{Peb}^{k\oplus}(G)$ (xor-ified). However, we can still form its $k\oplus$ -translation Q : the leaf clauses of Q are the 2^k clauses of the form $E' \in x_e^{k\oplus}$ and of the form $\overline{E'}, W$ for $\overline{E'} \in \overline{x}_e^{k\oplus}$. By choosing the appropriate left-right order for the hypotheses in Q , we arrange for $D = E$ to be the leftmost leaf clause of Q . Let $y = \rho(D)$. Then y is not one of the variables $x_{e,i}$, nor is y or \overline{y} in C^+ since condition (ii) does not hold for D . Therefore, we can modify Q by replacing $D = E$ with

$$\frac{D, \rho(D) \quad D, \overline{\rho(D)}}{D} \quad (6)$$

Form R_{i+1} from R_i by replacing C with the modified Q . This causes D to become learned as an input lemma in R_{i+1} . The other leaf clauses of Q all satisfy condition d. in R_{i+1} and thus become unfinished clauses of R_{i+1} :

they are all to the right of D . This adds fewer than 2^k new unfinished clauses to R_{i+1} .

Second suppose D equals $\overline{A}, \overline{B}, E$. For the moment, assume $e \neq w$. Consider the following derivation structure:

$$\frac{\frac{\frac{\overline{x}_a, \overline{x}_b, x_e}{\overline{x}_b, x_e} \quad x_a}{x_e} \quad x_b}{\overline{x}_e, W} \quad W \quad (7)$$

Define Q to be the $k\oplus$ -translation of (7), so Q is a tree-like derivation of the clause W from the clauses of the form $\overline{A}', \overline{B}', E'$, the form A' , the form B' , and the form \overline{E}', W , where the clauses $A', \overline{A}', B', \overline{B}', E'$, and \overline{E}' range over all members of $x_a^{k\oplus}, \overline{x}_a^{k\oplus}, x_b^{k\oplus}, \overline{x}_b^{k\oplus}, x_e^{k\oplus}$, and $\overline{x}_e^{k\oplus}$, respectively.

The left-right order of hypotheses in Q is chosen so that the clause D is the leftmost leaf clause of Q . The variable $y = \rho(D)$ is not one of the $x_{a,i}$'s, $x_{b,i}$'s, or $x_{e,i}$'s. Since condition (ii) does not hold, neither y nor \overline{y} is in C^+ . Therefore, we again replace the clause D in Q with the inference (6). Form R_{i+1} by replacing the unfinished clause C in R_i with the derivation Q . The inference (6) does not violate regularity, and causes D to be learned as an input lemma in R_{i+1} .

It is easy to check that the remaining leaf clauses of Q , which have the forms \overline{A}' , and \overline{B}' , and \overline{E}', W , and $\overline{A}', \overline{B}', E'$, satisfy condition d. These are thus valid as new unfinished clauses in R_{i+1} . A simple calculation shows that there are $2 \cdot 2^{3(k-1)} + 2^{2(k-1)} + 2^{k-1} < 2^{3k}$ many of these clauses.

On the other hand, suppose $e = w$. In this case, use the derivation structure

$$\frac{\frac{\overline{x}_a, \overline{x}_b, W}{\overline{x}_b, W} \quad x_a}{W} \quad x_b$$

instead of (7). Let Q be the $k\oplus$ -translation of this, and argue again as above. We leave the details to the reader.

This completes the construction of R_{i+1} in the case where C is just W . At least one new clause, D , from $\text{Peb}^{k\oplus}(G)$ has been learned as an input lemma. Since D is learned in the leftmost branch of Q , it is available for use as a learned clause for all future unfinished clauses. Fewer than 2^{3k} many new unfinished leaves have been introduced.

II. Now consider the case where C has the form \overline{U}, W . First suppose D is equal to $E \in x_e^\oplus$, where e is a source of G . We have $e \neq u$ since, even if

u is a leaf, P does not use the axiom of type (α) for U . Thus e and u are independent ancestors of w . Consider the derivation structure

$$\frac{x_e \quad \overline{x_e}, \overline{U}, W}{\overline{U}, W} \quad (8)$$

Let Q be the $k\oplus$ -translation of (8) arranged so that D is its leftmost leaf clause. Now argue as in the earlier cases to form R_{i+1} .

Second, suppose D is equal to $\overline{A}, \overline{B}, E$. We suppose that $e \neq w$, and leave the (slightly simpler) case of $e = w$ to the reader.² For the moment, suppose u is not equal to a or b . Again, it is not possible for u to equal e , since P does not use the clause of type (β) for u and its predecessors. Consider the derivation structure

$$\frac{\frac{\overline{x_a}, \overline{x_b}, x_e \quad ?_1, x_a}{?_1, \overline{x_b}, x_e} \quad ?_2, x_b}{\frac{?_1, ?_2, x_e}{\overline{U}, W} \quad ?_3, \overline{x_e}, W} \quad (9)$$

Here the subclauses $?_1, ?_2, ?_3$ are unknown and must be determined: in fact, one of them will be \overline{U} and the other two will be empty. The rules for determining these are as follows. First, if u is an ancestor of a (and thus not equal to a), select $?_1$ to equal \overline{U} . Second and otherwise, if u is an ancestor of b , select $?_2$ to equal \overline{U} . Otherwise, u and e are independent ancestors of w , so select $?_3$ to equal \overline{U} . For example, in the second case, the derivation (9) becomes

$$\frac{\frac{\overline{x_a}, \overline{x_b}, x_e \quad x_a}{\overline{x_b}, x_e} \quad \overline{U}, x_b}{\overline{U}, x_e \quad \overline{x_e}, W} \quad \overline{U}, W$$

Now let Q be the $k\oplus$ -translation of (9), with D the leftmost leaf clause of Q . Modify Q by replacing D with (6), and then form R_{i+1} by replacing C with Q . The other leaf clauses of Q have the form A' or \overline{U}, A' , the form B' or \overline{U}, B' , the form $\overline{E'}, W$ or $\overline{U}, \overline{E'}, W$, and the form $\overline{A'}, \overline{B'}, E'$. By examination of which variables are used for resolution in Q , it is clear that condition d. is satisfied for these clauses, and thus they are valid unfinished clauses in R_{i+1} .

This completes the description of R_{i+1} in this case. The clause D has become newly learned, and $< 2^{3k}$ many new unfinished clauses have been introduced.

²The idea is to modify (9) by omitting its final inference and changing the x_e 's to W 's.

Now suppose $u = a$, so $\overline{U} \in \overline{x}_a^{k\oplus}$. (The case where $u = b$ is similar.) In this case, the clause D has the form $\overline{U}, \overline{B}, E$ for some \overline{B} in $\overline{x}_b^{k\oplus}$ and some E in $x_e^{k\oplus}$. Let Q be the $k\oplus$ -translation of

$$\frac{\frac{\overline{U}, \overline{x}_b, x_e \quad x_b}{\overline{U}, x_e} \quad \overline{x}_e, W}{\overline{U}, W}$$

One of the leaf clauses in Q is equal to the clause D to be learned, and Q is ordered so that D is its leftmost leaf clause. As before, D is replaced in Q with (6) and then R_{i+1} is formed by replacing C in R_i with Q . The clause D has become learned as an input lemma, and $< 2^{2k}$ many new unfinished clauses have been introduced in R_{i+1} .

III. Third, consider the case where C has the form $\overline{U}, \overline{V}, W$. W.l.o.g., v is not an ancestor of u . First suppose D is $E \in x_e^{k\oplus}$ for e a source in G . We again have e is not equal to u, v , or w . Since also $e \in (G \upharpoonright w)[u, v]$ and since u and v are independent ancestors of w , there must exist a path from w to e that avoids u and v , a path from w to u that avoids e and v , and a path from w to v that avoids e and u . By looking at the point where these three paths first diverge, there is a vertex f that lies on exactly two of these paths. Suppose, for instance, that f lies on the two paths from w to u and to e . (The other two cases are similar.) Then u and e are independent ancestors of f , and f and v are independent ancestors of w . Form the derivation structure

$$\frac{x_e \quad \frac{\overline{U}, \overline{x}_e, x_f}{\overline{U}, x_f} \quad \overline{V}, \overline{x}_f, W}{\overline{U}, \overline{V}, W} \quad (10)$$

Let Q be the $k\oplus$ -translation of (10) with D as its leftmost clause and form R_{i+1} as before.

Second suppose D is $\overline{A}, \overline{B}, E$. We assume that $e \neq w$, leaving the slightly simpler $e = w$ case to the reader. (See the earlier footnote.) Suppose for the moment that u and v are distinct from a and b . As before, they cannot equal e . Consider the derivation structure

$$\frac{\frac{\overline{x}_a, \overline{x}_b, x_e \quad ?_1, x_a}{?_1, \overline{x}_b, x_e} \quad ?_2, x_b}{?_1, ?_2, x_e} \quad ?_3, \overline{x}_e, W}{\overline{U}, \overline{V}, W} \quad (11)$$

There are a number of subcases to consider; in each we describe how to set $?_1, ?_2$, and $?_3$. By default, $?_1, ?_2$, and $?_3$ (when not specified otherwise) are

to be the empty clause. Some of the subcases are overlapping, and when so, either option may be used. For example, it can happen that u is both an ancestor of e and a member of $(G \upharpoonright w)[e]$.

- Suppose u is an ancestor of e and $v \in (G \upharpoonright w)[e]$.
 - If u is an ancestor of a , set $?_1 := \overline{U}$ and $?_3 := \overline{V}$.
 - Otherwise, u is an ancestor of b , and we set $?_2 := \overline{U}$ and $?_3 := \overline{V}$.
- Suppose v is an ancestor of e , and $u \in (G \upharpoonright w)[e]$. This is handled like the previous case, interchanging \overline{U} and \overline{V} .
- Suppose neither u nor v is in $(G \upharpoonright w)[e]$.
 - If u is an ancestor of a and v an ancestor of b , set $?_1 := \overline{U}$ and $?_2 := \overline{V}$.
 - If u is an ancestor of b and v an ancestor of a , set $?_1 := \overline{V}$ and $?_2 := \overline{U}$.
 - If u and v are independent ancestors of a , set $?_1 := \overline{U}, \overline{V}$.
 - If u and v are independent ancestors of b , set $?_2 := \overline{U}, \overline{V}$.

These four subcases cover all possibilities since u and v are independent ancestors of w , and there is no path from w to either u or v which avoids e .

If any of the above subcases hold, form Q as the $k \oplus$ -translation of (11) with D as its leftmost clause, and form R_{i+1} by exactly the same construction as in the earlier cases.

If, however neither u nor v is an ancestor of e , then (11) cannot be used. Since also $e \in (G \upharpoonright w)[u, v]$, and u and v are independent ancestors of w , there must be a path from w to u that avoids v and e , a path from w to v that avoids u and e , and a path from w to e that avoids u and v . By looking at the point where these three paths first diverge, there is a vertex f that lies on exactly two of these paths. Suppose again that f lies on the two paths from w to u and to e . (The other two cases are similar.) Then, u and e are independent ancestors of f , and f and v are independent ancestors of w . Form the derivation structure

$$\begin{array}{c}
 \frac{\overline{x}_a, \overline{x}_b, x_e \quad x_a}{\overline{x}_b, x_e} \quad x_b \quad \overline{U}, \overline{x}_e, x_f \\
 \frac{\overline{x}_b, x_e \quad x_b}{x_e} \quad \overline{U}, \overline{x}_e, x_f \\
 \frac{\overline{U}, x_f \quad \overline{V}, \overline{x}_f, W}{\overline{U}, \overline{V}, W}
 \end{array} \tag{12}$$

Let Q be the $k\oplus$ -translation of (12) with D as its leftmost leaf clause. Replace D with (6), and then form R_{i+1} by replacing with C with Q . The other leaf clauses of Q including those of the form $\overline{V}, \overline{F}', W$ and of the form $\overline{U}, \overline{E}', F'$ become valid unfinished clauses that satisfy condition d. The clause D has become learned as an input lemma, and R_{i+1} has $< 2^{4k}$ new unfinished clauses.

We still have to consider the cases where u, v, a and b are not distinct. There are four (very similar) cases where only one of u and v is in $\{a, b\}$. For instance, suppose that $u = a$ and $v \neq b$. Then form Q as the $k\oplus$ -translation of

$$\frac{\frac{\overline{U}, \overline{x}_b, x_e \quad ?_1, x_b}{?_1, \overline{U}, x_e} \quad ?_2, \overline{x}_e, W}{\overline{U}, \overline{V}, W}$$

where $?_1 := V$ if v is an ancestor of b , and $?_2 := V$ if v is not an ancestor of b . Order Q so that D is its leftmost leaf clause, and form R_{i+1} as before.

In the case where $u = a$ and $v = b$, or vice-versa, let Q instead be the $k\oplus$ -translation of

$$\frac{\overline{U}, \overline{V}, x_e \quad \overline{x}_e, W}{\overline{U}, \overline{V}, W}$$

and proceed as before.

This completes case III. and the construction of R_{i+1} from R_i .

In cases (i)-(iii), an unfinished leaf is completely handled without adding a new unfinished clause. In cases (iv), a new $\text{Peb}^{k\oplus}(G)$ clause is learned as an input lemma while adding fewer than 2^{4k} many new unfinished leaves. There are fewer than $n2^{3(k-1)}$ many clauses that can be learned. Therefore the process of forming R_i 's terminates with a refutation R after a finite number of steps. The refutation R is a valid regRTI refutation of the $\text{GPeb}^{k\oplus}(G)$ clauses.

To estimate the size of R , note further that each time an unfinished leaf is handled by cases (i)-(iii) at most $k2^{3k}n^2$ many clauses are added. Therefore, the refutation R has size $O((2^{3k}n)(2^{4k})(k2^{3k}n^2) = O(2^{11k}n^3)$. The overall size of the $\text{GPeb}^{k\oplus}(G)$ clauses is $\Omega(2^{3k}n)$. Thus the size of R is polynomially bounded by the size of the $\text{GPeb}^{k\oplus}(G)$ principle; in fact, it is bounded by a degree four polynomial.

This completes the proof of Theorem 7. \square

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